

H-SPACES, LOOP SPACES AND THE SPACE OF POSITIVE SCALAR CURVATURE METRICS ON THE SPHERE

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ABSTRACT. For dimensions $n \geq 3$, we show that the space of metrics of positive scalar curvature on the sphere S^n , denoted $\mathcal{Riem}^+(S^n)$, is homotopy equivalent to a subspace which takes the form of a H -space with a homotopy commutative, homotopy associative product operation. This product operation is based on the connected sum construction. We then exhibit an action of the little n -disks operad on this subspace which, using results of Boardman, Vogt and May implies that when $n = 3$ or $n \geq 5$, $\mathcal{Riem}^+(S^n)$ is weakly homotopy equivalent to an n -fold loop space.

1. INTRODUCTION

This work is motivated by the problem of understanding the topology of the space of metrics of positive scalar curvature (*psc-metrics*) on the sphere S^n . This space is denoted $\mathcal{Riem}^+(S^n)$ and is an open subspace of the space of all Riemannian metrics on S^n , $\mathcal{Riem}(S^n)$, equipped with its standard smooth topology. It is known that when $n = 2$, the space $\mathcal{Riem}^+(S^n)$ is contractible; see [13]. When $n = 3$, we know from a recent result of Marques that this space is path connected; see [10]. In fact it is thought by experts that the space is contractible in this case also. When $n \geq 4$ however, the space $\mathcal{Riem}^+(S^n)$ is usually not path connected; see for example [2]. Furthermore, the groups $\pi_k(\mathcal{Riem}^+(S^n))$ are often non-trivial; see [3] and [6]. In this paper we make the following contribution.

Main Results.

- (i.) *When $n \geq 3$, the space $\mathcal{Riem}^+(S^n)$ is homotopy equivalent to a subspace which admits a homotopy product (i.e. is a H -space). Furthermore this product is homotopy commutative and homotopy associative.*
- (ii.) *When $n = 3$ or $n \geq 5$, the space $\mathcal{Riem}^+(S^n)$ is weakly homotopy equivalent to an n -fold loop space.*

Definitions of the terms H -space and Loop Space are given in section 2.2. We will not discuss the topological implications of such structure on $\mathcal{Riem}^+(S^n)$ other than to point out that the condition that a topological space is a H -space, or especially an iterated loop space, imposes significant restrictions on its homotopy type. For more on this, see chapter 4 of [9].

The main idea is as follows. We specify a subspace $\mathcal{Riem}_{\text{cap}(p_0)}^+(S^n) \subset \mathcal{Riem}^+(S^n)$ consisting of psc-metrics which take the form of a “round cap” near a fixed base point $p_0 \in S^n$. It is known from results in [17] that this subspace is homotopy equivalent to the space $\mathcal{Riem}^+(S^n)$. We then construct a product on this subspace. Roughly speaking, this product involves removing the caps of the factor metrics and then taking a connected sum via some appropriate connecting metric. There is one important caveat. We need to ensure that the metric obtained by this product has a base point, something which the individual factors lose once we remove the caps. Hence we use, as an intermediary metric, a psc-metric on the

sphere containing 3 such caps, two of which will be removed for the attachments and the third which will be a base point cap. A specific case of this 3-capped metric, which involves pushing out small caps on a round sphere (the so-called *mine metric*), is later generalised to a family of such metrics with multiple caps. This allows us to realise a positive scalar curvature analogue of the *operad of little n -dimensional disks*; see below for a description of this object. After exhibiting an action of the operad on the space $\mathcal{Riem}_{\text{cap}(p_0)}^+(S^n)$, we make use of a theorem of Boardman, Vogt and May [9] to demonstrate that this space is weakly homotopic to an n -fold loop space when $n = 3$ or $n \geq 5$. The proof of this result also makes use of a recent theorem by Botvinnik, Theorem B. of [1], concerning isotopy and concordance of psc-metrics. The hypothesis that $n = 3$ or $n \geq 5$ stems from the fact that Botvinnik's theorem is false when $n = 4$. Whether or not the same is true of our result is unknown.

1.1. Organisation of the paper. The paper is organised as follows. After taking care of some notational and technical issues in section 2, we go on in section 3 to describe certain spaces of psc-metrics on the disk D^n and on the cylinder $I \times S^{n-1}$. This is done in 3.1 and 3.2 where we respectively describe the spaces of infinitesimal *torpedo* metrics $\mathcal{T}_{\mathcal{Riem}}^+$ and *connecting* metrics $\mathcal{C}_{\mathcal{Riem}}^+$. The former space will be used to describe various kinds of “caps” on the sphere metrics while the elements of the latter space will be used to glue metrics together. In 3.3, we describe such an *elementary connecting construction* in the form a continuous map. This elementary construction plays an important role later on in defining the homotopy product.

In section 4 we prove the first of our main results: the *H-Space Theorem*. We begin in 4.1 by defining the *space of psc-metrics with caps*, $\mathcal{Riem}_{\text{cap}(\mathbf{p})}^+(S^n)$ where \mathbf{p} is a finite collection $\{p_0, p_1, \dots, p_k\}$, of points on S^n . This is the space of psc-metrics on S^n which have disjoint caps at each of the points p_i . In 4.2, we generalise the elementary connecting construction from section 3.3 to describe a method of joining pairs of elements in such spaces to form new psc-metrics on S^n . Thus, rather than gluing together two elements of $\mathcal{Riem}_{\text{cap}(p_0)}^+(S^n)$ to end up with an element of $\mathcal{Riem}^+(S^n)$ which lacks a base point and a cap, we use an intermediary metric g_3 in $\mathcal{Riem}_{\text{cap}(\mathbf{p})}^+(S^n)$ where \mathbf{p} consists of 3 distinct points p_0, p_1 and p_2 . The idea is to remove the caps at p_1 and p_2 to attach pairs of metrics to g_3 resulting in a metric with base point p_0 untouched. This process is realised as a continuous map at the end of 4.2. In 4.3, we prove our first main result, Theorem 4.7, in which we show that this map defines a homotopy product on $\mathcal{Riem}_{\text{cap}(p_0)}^+(S^n)$ which is homotopy commutative and homotopy associative. A minor consequence, Corollary 4.8, is that the fundamental group of $\mathcal{Riem}^+(S^n)$, with base point the standard round metric, is Abelian.

The final section, section 5, concerns the second main result: the *Loop Space Theorem*. In 5.1, we recall the *operad of little n -dimensional disks* denoted \mathbb{D}_n , before describing the notion of a \mathbb{D}_n -space. Roughly, this is a topological space which admits a type of action of \mathbb{D}_n . We then recall a theorem of Boardman, Vogt and May which states that a \mathbb{D}_n -space is weakly homotopy equivalent to an n -fold loop space provided the set of its path components π_0 forms a group. Sections 5.2 and 5.3 are then devoted to showing that $\mathcal{Riem}_{\text{cap}(p_0)}^+(S^n)$ satisfies the hypotheses of this theorem. In 5.2 we demonstrate that $\mathcal{Riem}_{\text{cap}(p_0)}^+(S^n)$ is a \mathbb{D}_n -space. The main work here is defining the action, involving an “embedding” of \mathbb{D}_n into $\mathcal{Riem}_{\text{cap}(p_0)}^+(S^n)$. In section 5.3 we tackle the problem of showing that $\pi_0(\mathcal{Riem}_{\text{cap}(p_0)}^+(S^n))$

is a group under the operation induced by connected sum. This involves a brief discussion of the notions of psc-isotopy and psc-concordance and makes use of results by Botvinnik [1], Gajer [4] and Marques [10]. Finally, in 5.4, we combine the work of the previous two sections and formally prove Theorem 5.5, our second main result.

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2. PRELIMINARY DETAILS

2.1. Manifolds and Metrics. Let X be a smooth n -dimensional manifold X , possibly with non-empty boundary. In this paper, the dimension n is assumed always to be greater than or equal to 3. We denote by $\mathcal{Riem}(X)$ the space of Riemannian metrics on X equipped with its standard C^∞ -topology; see section 1.1 of [15] for a description. Contained inside $\mathcal{Riem}(X)$ as an open subspace is the space of psc-metrics on X , denoted $\mathcal{Riem}^+(X)$. In the case when $\partial X \neq \emptyset$ it is common to consider only a subspace of $\mathcal{Riem}^+(X)$ of metrics which satisfy some constraint near the boundary. We will need such a constraint in this paper also and will return to this issue shortly.

We denote by D^n and S^n the standard smooth disk and sphere of dimension n and by ds_n^2 , the standard round metric of radius 1 on S^n . As smooth topological objects, we model the disk $D^n = D^n(1)$ as the set of points $\{x \in \mathbb{R}^n : |x| \leq 1\}$ and sphere $S^n = S^n(1)$ as the set $\{x \in \mathbb{R}^{n+1} : |x| = 1\}$. In constructing metrics on these spaces, we will often work on an underlying disk or sphere $D^n(r)$ or $S^n(r)$ where the radius $r \neq 1$. The re-scaling function $x \mapsto rx$ gives a canonical way of pulling back metrics to the standard disk or sphere. Thus, we will often declare a metric which has been constructed on a general $D^n(r)$ or $S^n(r)$ to be a metric on D^n or S^n assuming the metric to be pulled back in this way. Similarly, all metrics constructed on $[0, b] \times S^{n-1}$ when $b > 0$ should be considered metrics on $I \times S^{n-1}$ (where $I = [0, 1]$) via the obvious pull-back.

We now return to the question of boundary conditions on certain metrics. In this paper, the only manifolds with boundary we will encounter are the disk D^n and the cylinder $I \times S^{n-1}$. Our various constructions will involve attaching Riemannian disks along their boundary or via some intermediary cylinder metric to obtain new metrics on the sphere. We need to ensure smooth attachment at the metric level. One way to do this is to restrict ourselves to working with metrics which take the structure of a standard round cylinder near the boundary, at least infinitesimally. With this in mind we specify subspaces $\mathcal{Riem}_0^+(D^n) \subset \mathcal{Riem}^+(D^n)$ and $\mathcal{Riem}_0^+(I \times S^{n-1}) \subset \mathcal{Riem}^+(I \times S^{n-1})$ as follows. Beginning with the disk D^n , fix some $\epsilon > 0$ and consider $D^n = D^n(1)$ as a submanifold of the disk of radius $1 + \epsilon$, $D^n(1 + \epsilon)$. Let t denote the radial distance from the origin. Furthermore let $\text{Ann}(1, 1 + \epsilon)$ denote the closure of annulus $D^n(1 + \epsilon) \setminus D^n(1)$. Let $\mathcal{Riem}_\epsilon^+(D^n(1 + \epsilon))$ denote the space of psc-metrics on $D^n(1 + \epsilon)$ defined as follows:

$$\mathcal{Riem}_\epsilon^+(D^n(1 + \epsilon)) := \{g \in \mathcal{Riem}^+(D^n(1 + \epsilon)) : g|_{\text{Ann}(1, 1 + \epsilon)} = dt^2 + \delta^2 ds_{n-1}^2 \text{ for some } \delta > 0\}.$$

We next consider the restriction map:

$$\begin{aligned}\mathcal{Riem}_\epsilon^+(D^n(1+\epsilon)) &\longrightarrow \mathcal{Riem}^+(D^n(1)), \\ g &\longmapsto g|_{D^n(1)}.\end{aligned}$$

Finally, we define the space $\mathcal{Riem}_0^+(D^n)$ to be the image of this restriction map. We similarly define $\mathcal{Riem}_\epsilon^+([- \epsilon, 1 + \epsilon] \times S^{n-1})$ as the space of psc-metrics on $[- \epsilon, 1 + \epsilon] \times S^{n-1}$ which take a product structure $dt^2 + \delta_0^2 ds_{n-1}^2$ on $[- \epsilon, 0] \times S^{n-1}$ and $dt^2 + \delta_1^2 ds_{n-1}^2$ on $[0, 1 + \epsilon] \times S^{n-1}$ for any $\delta_0, \delta_1 > 0$. The space $\mathcal{Riem}_0^+(I \times S^{n-1})$ is then the image of the analogous restriction map into $\mathcal{Riem}^+(I \times S^{n-1})$.

2.2. H -Spaces and Loop spaces. A topological space Z is a H -space if Z is equipped with a continuous multiplication map $\mu : Z \times Z \rightarrow Z$ and an identity element $e \in Z$ so that the maps from Z to Z given by $x \mapsto \mu(x, e)$ and $x \mapsto \mu(e, x)$ are both homotopy equivalent to the identity map $x \mapsto x$. There are stronger versions of this definition where the above homotopies to the identity map are required to be homotopic through pointed maps $(Z, e) \rightarrow (Z, e)$ or where multiplication by the identity is the identity map. It is well known that in the case when Z is homotopy equivalent to a CW complex, Z admits a product which agrees with one of these definitions if and only if it admits products agreeing with the other two; see chapter 3.C of [7]. Moreover, it follows from the work of Palais in [11] that for any smooth compact manifold X , $\mathcal{Riem}^+(X)$ is homotopy equivalent to a CW complex. Thus we feel justified in using the weaker definition. A H -space Z is said to be homotopy commutative if the maps μ and $\mu \circ \omega$, where $\omega : Z \times Z \rightarrow Z \times Z$ is the “flip” map defined $\omega(x, y) = (y, x)$, are homotopy equivalent. Finally, Z is a homotopy associative H -space if the maps from $Z \times Z \times Z$ to Z given by $(x, y, z) \mapsto \mu(\mu(x, y), z)$ and $(x, y, z) \mapsto \mu(x, \mu(y, z))$ are homotopy equivalent.

Given a topological space Z with a prescribed base point $z_0 \in Z$, we may consider the space of all loops based at z_0 . This is the space of all continuous maps $\gamma : [0, 1] \rightarrow Z$ so that $\gamma(0) = \gamma(1) = z_0$. This space is known as the *loop space of Z* , denoted $\Omega(Z, z_0)$, with base point the constant loop at z_0 . Assuming the base point to be understood, we simply write ΩZ . Repeated application of this construction yields the k -th iterated loop space $\Omega^k Z = \Omega(\Omega \cdots (\Omega Z))$ where at each stage the new base point is simply the constant loop at the old base point. We close by pointing out that a loop space is also a H -space with the multiplication determined by concatenation of loops. Whether or not a given H -space has the structure of a loop space is a more complicated problem concerning certain “coherence” conditions on the homotopy associativity of the multiplication. It is a theorem of Stasheff that a space satisfies these conditions, is a so-called A_∞ -space, if and only if it is a loop space; see Theorem 4.18 in [9]. The notion of an *operad* was constructed to more efficiently describe these coherence conditions; something we will return to in section 5.

3. INFINITESIMAL TORPEDO METRICS AND CONNECTING METRICS

3.1. Infinitesimal Torpedo Metrics. We will construct a particular family of psc-metrics on the disk D^n known as infinitesimal torpedo metrics. Recall that a torpedo metric on the disk D^n is an $O(n)$ -symmetric metric which is round near the centre of the disk but transitions to a standard round Riemannian cylinder (neck) $[0, \epsilon] \times S^{n-1}$ near the boundary. For a detailed discussion of these metrics and their variants, see chapter 1 of [15]. Roughly speaking an infinitesimal torpedo metric takes this product structure only infinitesimally at

the boundary of D^n . Importantly however, it smoothly attaches along the boundary to an end of a round cylinder $[0, b] \times S^{n-1}$. For each $\delta > 0$, let \mathcal{T}_δ be the set of smooth functions $\eta : [0, \frac{\pi}{2}\delta] \rightarrow [0, \infty)$ which satisfy the following requirements:

- (i) $\eta(t) = \delta \sin \frac{t}{\delta}$ when t is near 0,
- (ii) $\eta(\frac{\pi}{2}\delta) = \delta$,
- (iii) $\eta''(t) < 0$ when $0 \leq t < \frac{\pi}{2}$,
- (iv) $\eta_-^{(k)}(\frac{\pi}{2}\delta) = 0$ for all $k \geq 1$,

where $\eta_-^{(k)}$ represents the left sided k -th derivative of η . Essentially we want functions which behave like \sin for the most part but end with all zero derivatives, as illustrated in Figure 1 below. We may think of the set \mathcal{T}_δ^+ as a space under the usual C^∞ -function space topology. See Chapter 2 of [8] for details of such topologies.

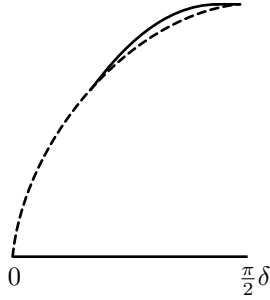


FIGURE 1. Comparing the graph of η with the graph of the standard \sin function represented by the dashed curve

Given any function $\eta \in \mathcal{T}_\delta^+$, the formula:

$$(3.1) \quad g^\eta = dt^2 + \eta^2(t) ds_{n-1}^2,$$

where ds_{n-1}^2 is the standard round metric of radius 1 on the sphere S^{n-1} , restricts to a Riemannian metric on $(0, \frac{\pi}{2}] \times S^{n-1}$. It also closes in at the end $\{0\} \times S^{n-1}$ in such a way as to uniquely determine a smooth Riemannian metric on the disk D^n . This follows from the results of Chapter 1, Section 3.4 of [12].

Remark 3.1. *We will intermittently regard g^η as a metric on both $[0, \delta\frac{\pi}{2}] \times S^{n-1}$ and on D^n depending on our circumstances.*

The scalar curvature R of the metric g^η at the point $(t, \theta) \in (0, \frac{\pi}{2}] \times S^{n-1}$ is given by the formula:

$$(3.2) \quad R(t, \theta) = -2(n-1) \frac{\eta''(t)}{\eta(t)} + (n-1)(n-2) \frac{1 - (\eta'(t))^2}{\eta(t)^2}.$$

It is clear from this formula and the conditions on the second derivative of η that the scalar curvature of g^η is always positive. (Recall we assume $n \geq 3$. When $n = 2$ the best we can say is that $R \geq 0$.) We then obtain a space of metrics $\mathcal{T}_{\text{Riem}}^+(\delta)$, the subspace of $\text{Riem}_0^+(D^n)$ defined:

$$\mathcal{T}_{\text{Riem}}^+(\delta) := \{g^\eta \in \text{Riem}_0^+(D^n) : \eta \in \mathcal{T}_\delta^+\},$$

where g^η is given by formula 3.1 above on $(0, \frac{\pi}{2}] \times S^{n-1}$ but of course extends uniquely onto D^n . Importantly, metrics in $\mathcal{T}_{\mathcal{Riem}}^+(\delta)$ all behave the same way at the boundary. For our purposes we will require δ to vary and so we define the space of functions \mathcal{T}^+ and subspace of psc-metrics $\mathcal{T}_{\mathcal{Riem}}^+ \subset \mathcal{Riem}_0^+(D^n)$ as follows:

$$\mathcal{T}^+ = \bigcup_{\delta > 0} \mathcal{T}_\delta^+ \text{ and } \mathcal{T}_{\mathcal{Riem}}^+ = \bigcup_{\delta > 0} \mathcal{T}_{\mathcal{Riem}}^+(\delta).$$

Lemma 3.1. *When $n \geq 3$ the space $\mathcal{T}_{\mathcal{Riem}}^+$ is a path connected space.*

Proof. It suffices to show that \mathcal{T}^+ is path connected. It is elementary to check that for each $\delta > 0$, the space \mathcal{T}_δ^+ is convex and thus path connected. Furthermore, for any $\eta \in \mathcal{T}_\delta^+$, the formula:

$$\eta_s(t) = \frac{1}{s}\eta(st),$$

where $s > 0$, describes an element of $\mathcal{T}_{s\delta}^+$. This allows us to continuously adjust η through elements of \mathcal{T}^+ and end in a subset $\mathcal{T}_{s\delta}^+$ of our choosing. \square

3.2. Connecting Metrics. We will now define a particular space of warped product metrics on the cylinder $[0, b] \times S^{n-1}$, where $b \geq 1$. Let α denote a standard smooth cut-off function $[0, 1] \rightarrow [0, 1]$ which satisfies the following conditions:

- (i) $\alpha(0) = 0$ and $\alpha(1) = 1$,
- (ii) $\alpha'(t) \geq 0$,
- (iii) $\alpha_+^{(k)}(0) = \alpha_-^{(k)}(b) = 0$ for all $k \geq 1$,

where $\alpha_+^{(k)}$ and $\alpha_-^{(k)}$ denote right and left sided k -th derivatives. More generally we define a continuous family of functions $\alpha_{\lambda b} : [0, b] \rightarrow [0, \infty)$, by the formula:

$$\alpha_{\lambda b}(t) = \lambda \alpha\left(\frac{t}{b}\right),$$

where $\lambda \geq 0$ and $b \geq 1$. We now make one further addition to this family. We denote by \mathcal{C} , the family of *connecting functions*:

$$\mathcal{C} := \{\beta = \alpha_{\lambda b} + \delta : \lambda, b \in [0, \infty) \text{ and } \delta > 0\}.$$

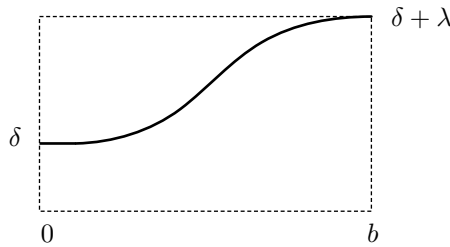


FIGURE 2. The graph of a function in \mathcal{C}

This family forms a path-connected subspace of the space obtained by taking the union over all $b \geq 0$ of the spaces of smooth functions $[0, b] \rightarrow [0, \infty)$, each considered under the C^∞ -topology. Each function $\beta \in \mathcal{C}$ gives rise to the following warped product metric g^β on a cylinder $[0, b] \times S^{n-1}$:

$$g^\beta = dt^2 + \beta^2(t)ds_{n-1}^2.$$

We are especially interested in the case when such a metric has positive scalar curvature. We will shortly specify a subspace of \mathcal{C} which will consist of metrics which will act as connecting metrics in our construction. Before this we make an elementary observation in the form of a lemma.

Lemma 3.2. *Let $n \geq 3$, $\delta > 0$ and $\lambda \geq 0$. The scalar curvature of the metric g^β , where $\beta = \alpha_{\lambda b} + \delta$, is positive for some sufficiently large $b > 0$.*

Proof. This is actually an elementary case of a well known result from [5]. The scalar curvature R for this metric at the point $(t, \theta) \in (0, b) \times S^{n-1}$ is given by formula 3.2 with η replaced by β . Further expanding it we obtain:

$$(3.3) \quad R(t, \theta) = \frac{1}{\lambda\alpha(\frac{t}{b}) + \delta} \left[-2(n-1)\frac{\lambda}{b^2}\alpha''\left(\frac{t}{b}\right) + (n-1)(n-2)\frac{1 - (\frac{\lambda}{b}\alpha'(\frac{t}{b}))^2}{\lambda\alpha(\frac{t}{b}) + \delta} \right].$$

The quantity λ is fixed and α, α' and α'' are all bounded. As b increases, the second summand grows positively while the absolute value of the first summand tends to zero. \square

Of course, the extent to which b may need to be increased is also affected by the size of both λ and δ . Obviously if λ is zero, any b will work. Furthermore, small values of δ will mean smaller values of b are sufficient. However for $\lambda > 0$, some positivity of the second derivative of β is inevitable and a sufficiently large b must be chosen. It will be important for us to have a way of choosing such a b . Notice from formula 3.3 that as b grows, the scalar curvature tends towards a quantity which is bounded below by $\frac{1}{(\lambda+\delta)^2}$. With this in mind, we define for each pair $\delta > 0, \lambda \geq 0$, the quantity $c = c(\delta, \lambda)$ by the formula:

$$c(\delta, \lambda) = \frac{1}{2(\delta + \lambda)^2}$$

We then define the quantity $\bar{b} = \bar{b}(\delta, \lambda)$ as follows:

$$\bar{b}(\delta, \lambda) = \inf\{b \in [1, \infty) : R(t, \theta) \geq c(\delta, \lambda) \text{ for all } (t, \theta) \in [0, b] \times S^{n-1}\},$$

where R is the scalar curvature of the metric g^β and $\beta = \alpha_{\lambda b} + \delta$. Hence, for each pair $(\delta, \lambda) \in (0, \infty) \times [0, \infty)$, there is a unique smooth function in \mathcal{C} , denoted $\tau(\delta, \lambda)$ and defined:

$$\begin{aligned} \tau(\delta, \lambda) : [0, \bar{b}] &\rightarrow [0, \infty), \\ t &\mapsto \lambda\alpha(t\bar{b}^{-1}) + \delta. \end{aligned}$$

This association is best thought of as a continuous map:

$$\begin{aligned} (0, \infty) \times [0, \infty) &\longrightarrow \mathcal{C}, \\ (\delta, \lambda) &\longmapsto \tau(\delta, \lambda). \end{aligned}$$

We define the *space of connecting functions* to be the image of this map in \mathcal{C} and denote it by \mathcal{C}^+ . Each element $\tau \in \mathcal{C}^+$ determines a psc-metric $g^\tau \in \mathcal{Riem}_0^+(I \times S^{n-1})$. We denote by $\mathcal{C}_{\mathcal{Riem}}^+$ the space defined:

$$\mathcal{C}_{\mathcal{Riem}}^+ = \{g \in \mathcal{Riem}_0^+(I \times S^{n-1}) : g = g^\tau \text{ for some } \tau \in \mathcal{C}^+\}.$$

This is the *space of connecting metrics*.

3.3. An Elementary Connecting Construction. We close this section with an elementary construction. Given a pair of metrics $g, h \in \mathcal{Riem}_0^+(D^n)$, there is an obvious way to construct a psc-metric on the sphere S^n by means of an appropriate connecting metric. Specifically, suppose g and h have respective boundary spheres of radii δ_0 and δ_1 . Without loss of generality we assume that $\delta_0 \leq \delta_1$. The pair $\delta_0, \lambda = \delta_1 - \delta_0$ now determine a unique connecting metric $g^{\tau(\delta_0, \lambda)} \in \mathcal{C}_{\mathcal{Riem}}^+$. The disks then smoothly attach at appropriate ends of the cylinder to form the psc-metric $g = g \cup g^\tau \cup h$ in $\mathcal{Riem}^+(S^n)$ as shown in Figure 3.

Remark 3.2. *Strictly speaking this metric is not yet a metric on S^n however it can be pulled back to a metric on the standard sphere S^n in a canonical way. Essentially, we fix a belt neighborhood around the equator which is identified with the cylinder $I \times S^{n-1}$. We then extend this to a diffeomorphism which we use throughout the construction.*

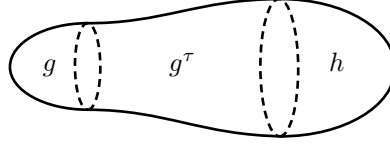


FIGURE 3. The sphere metric formed as a union of g , g^τ and h .

We will summarize the above construction in the form of a continuous map as follows. Firstly, let ρ be the *radius measuring map* defined:

$$(3.4) \quad \begin{aligned} \rho : \mathcal{Riem}_0^+(D^n) &\longrightarrow (0, \infty), \\ g &\longmapsto \rho(g) = \text{Radius of sphere } g|_{\partial D^n}. \end{aligned}$$

Then let \mathcal{P} denote the subspace of $\mathcal{Riem}_0^+(D^n) \times \mathcal{Riem}_0^+(D^n)$ defined:

$$\mathcal{P} = \{(g, h) \in \mathcal{Riem}_0^+(D^n) \times \mathcal{Riem}_0^+(D^n) : \rho(g) \leq \rho(h)\}$$

We then define the *joining map* J as:

$$(3.5) \quad \begin{aligned} J : \mathcal{P} &\longrightarrow \mathcal{Riem}^+(S^n) \\ (g, h) &\longmapsto g \cup g^{\tau(\rho(g), \rho(h) - \rho(g))} \cup h, \end{aligned}$$

where the output metric is constructed precisely as we described above. Furthermore, let ω denote the *ordering map* from $\mathcal{Riem}_0^+(D^n) \times \mathcal{Riem}_0^+(D^n)$ to \mathcal{P} , which sends a pair (g, h) to the pair $\omega(g, h)$ defined:

$$(3.6) \quad \omega(g, h) = \begin{cases} (g, h) & \text{if } \rho(g) \leq \rho(h), \\ (h, g) & \text{if } \rho(g) > \rho(h). \end{cases}$$

Proposition 3.3. *The composition map $J \circ \omega : \mathcal{Riem}_0^+(D^n) \times \mathcal{Riem}_0^+(D^n) \rightarrow \mathcal{Riem}^+(S^n)$ is a continuous map.*

Proof. This is immediate from the construction. □

4. FROM CONNECTED SUMS TO HOMOTOPY PRODUCTS

4.1. The space of psc-metrics with caps on S^n . We will now consider a slightly more detailed construction which utilises the one at the end of the previous section. When studying spaces of metrics on a manifold one often fixes a particular metric, called a reference metric, to be used to unambiguously specify coordinate balls or an exponential map. Although in principle the choice of reference metric does not matter, it is convenient in our case to use the standard round metric of radius 1, ds_n^2 as a reference metric on S^n . Let p_0 be any point in S^n . A choice of orthonormal (with respect to ds_n^2) basis for the tangent space $T_{p_0}S^n$ to S^n at p_0 gives rise to an isomorphism from \mathbb{R}^n to $T_{p_0}S^n$. Composing this with the exponential map gives rise to a smooth pointed map $(\mathbb{R}^n, 0) \rightarrow (S^n, p_0)$ which restricts to an embedding on small disks around $0 \in \mathbb{R}^n$. By precomposing with an appropriate rescaling we obtain an embedding of the standard unit disk D^n into S^n . We will call this embedding ϕ_{p_0} and let D_{p_0} denote its image in S^n . Finally, let p'_0 denote the antipodal point to p_0 on S^n , let $D'_{p_0} = \text{closure}(S^n \setminus D_{p_0})$ and let $\phi'_{p_0} : D^n \rightarrow D'_{p_0}$ denote the corresponding complementary embedding obtained from an appropriate restriction of the exponential map at p'_0 . We now define a subspace of $\mathcal{Riem}^+(S^n)$, which we denote $\mathcal{Riem}^+_{\text{cap}(p_0)}(S^n)$ as follows:

$$\mathcal{Riem}^+_{\text{cap}(p_0)}(S^n) = \{g \in \mathcal{Riem}^+(S^n) : \phi_{p_0}^*(g|_{D_0}) \in \mathcal{T}_{\mathcal{Riem}}^+\}.$$

Furthermore there is a restriction map:

$$\begin{aligned} \mathcal{Riem}^+_{\text{cap}(p_0)}(S^n) &\longrightarrow \mathcal{Riem}^+(D^n) \\ g &\longmapsto (\phi'_{p_0})^*(g|_{D'_{p_0}}). \end{aligned}$$

Thus each element of $\mathcal{Riem}^+_{\text{cap}(p_0)}(S^n)$ is a metric which has a sort of round “cap” at the point p_0 . This is the space of *psc-metrics with caps at p_0* . More generally, if $\mathbf{p} = \{p_0, p_1, p_2, \dots, p_k\} \subset S^n$ is a finite collection of points on S^n , we may specify around each of these points, closed disjoint normal coordinate neighbourhoods $D_{p_0}, D_{p_1}, \dots, D_{p_k}$ of the type described above with corresponding diffeomorphisms $\phi_{p_i} : D^n \rightarrow D_{p_i}$ and complementary diffeomorphisms $\phi'_{p_i} : D^n \rightarrow D'_{p_i}$ for each $i \in \{0, 1, 2, \dots, k\}$. We now define the space $\mathcal{Riem}^+_{\text{cap}(\mathbf{p})}(S^n)$ as follows:

$$\mathcal{Riem}^+_{\text{cap}(\mathbf{p})}(S^n) = \bigcap_{i=0}^k \mathcal{Riem}^+_{\text{cap}(p_i)}(S^n).$$

In Figure 4 below we represent an element of $\mathcal{Riem}^+_{\text{cap}(\mathbf{p})}(S^n)$ where $\mathbf{p} = \{p_0, p_1, p_2, p_3\}$ is a set of four distinct points on S^n .

We now make a number of important observations about the space $\mathcal{Riem}^+_{\text{cap}(\mathbf{p})}(S^n)$. Firstly, the choice of orthonormal basis for each tangent space is unimportant.

Lemma 4.1. *For $n \geq 3$, the subspace $\mathcal{Riem}^+_{\text{cap}(\mathbf{p})}(S^n) \subset \mathcal{Riem}^+(S^n)$ remains fixed if we vary the choice of orthonormal basis for $T_{p_i}S^n$ for each $p_i \in \mathbf{p}$.*

Proof. This essentially follows from the rotational symmetry of the caps. For a detailed proof, see [16]. \square

For a single point p , the topology of $\mathcal{Riem}^+_{\text{cap}(p)}(S^n)$ is also unaffected by the choice of p . In particular we have the following lemma.

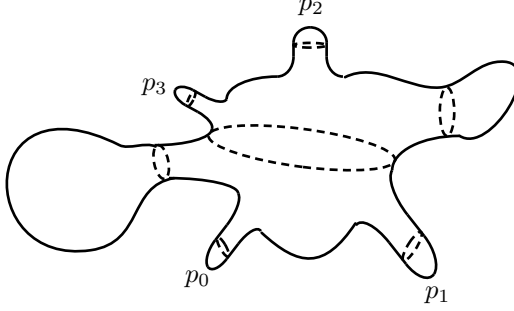


FIGURE 4. A sphere with four caps.

Lemma 4.2. *For $n \geq 3$ and for any $p, q \in S^n$, the spaces $\mathcal{Riem}_{\text{cap}(p)}^+(S^n)$ and $\mathcal{Riem}_{\text{cap}(q)}^+(S^n)$ are homeomorphic.*

Proof. Let $\text{ro}_{(p,q)}$ denote the rotation from p to q along the great circle (with respect to the standard round metric) containing p and q . Then the map which sends a metric g on S^n to the pull back metric $\text{ro}_{(p,q)}^* g$ defines a homeomorphism from $\mathcal{Riem}_{\text{cap}(q)}^+(S^n)$ to $\mathcal{Riem}_{\text{cap}(p)}^+(S^n)$. \square

Finally, we recall that these spaces are all homotopy equivalent to the space of all psc-metrics on S^n . This is a consequence of the following lemma based on a famous construction of Gromov and Lawson in [5].

Lemma 4.3. *Let M^n be a compact smooth manifold of dimension $n \geq 3$. Suppose $g_t, t \in K$ is a compact family of psc-metrics on M , indexed by a compact space K . Suppose also that $p_s, s \in I$ is a path in M . Then there is a continuous family of neighbourhoods U_s of p_s in M as s varies along I , a constant $\delta > 0$ and a compact family of psc-metrics $g'_{st}, (s, t) \in I \times K$ which satisfies the following:*

- (1) *For each $(s, t) \in I \times K$, $g'_{st} = g_t$ outside of U_s .*
- (2) *For $s \in I$, there is a continuously varying family of neighbourhoods D_s of p_s , with $D_s \subset U_s$ so that for each $(s, t) \in I \times K$, g'_{st} takes the form of an infinitesimal torpedo metric of radius δ on D_s .*
- (3) *For each $s_0 \in I$ there is a continuous homotopy through families of psc-metrics on S^n which deforms the family $\{g_t : t \in K\}$ into $\{g'_{s_0 t} : t \in K\}$.*

Proof. This is a special case of Theorem 2.13 from [15]. \square

In the case when K is a point and p_s is the constant path, parts (1) and (2) imply that the connected sum of manifolds (of dimension $n \geq 3$) which admit psc-metrics admits a psc-metric. This is a special case of the well-known Surgery Theorem due to Gromov and Lawson [5] and proved independently by Schoen and Yau [14]. The following application of the above fact is proved in [17] and will be of interest to us.

Lemma 4.4. *When $n \geq 3$ and \mathbf{p} is a finite collection of points on the sphere S^n , the spaces $\mathcal{Riem}_{\text{cap}(\mathbf{p})}^+(S^n)$ and $\mathcal{Riem}^+(S^n)$ are homotopy equivalent.*

Proof. This is just a special case of Lemma 3.3 of [17]. \square

4.2. A Product Construction. We now return to the “connecting” construction described in the previous section. For a given point $p_0 \in S^n$, we wish to define a homotopy product on $\mathcal{Riem}_{\text{cap}(p_0)}^+(S^n)$. Naively, one might consider taking two psc-metrics with caps, removing the caps and then gluing them together with the uniquely specified connecting metric. Unfortunately this produces a metric on a sphere with no base point and so a slightly more intricate multiplication is required. Before describing the more intricate construction, it is worth describing a version of this naive construction, as it gets us most of the way there.

Let $\mathbf{p} = \{p_0, p_1, \dots, p_k\}$ and $\mathbf{q} = \{q_0, q_1, \dots, q_l\}$ be two finite sets of points on S^n . We will assume that $p_i \neq p_j$ when $i \neq j$ and that $q_i \neq q_j$ when $i \neq j$ but make no assumptions about whether or not $p_i = q_j$. We now consider the corresponding spaces $\mathcal{Riem}_{\text{cap}(\mathbf{p})}^+(S^n)$ and $\mathcal{Riem}_{\text{cap}(\mathbf{q})}^+(S^n)$. For each pair of integers (i, j) with $0 \leq i \leq k$ and $0 \leq j \leq l$ we define the ij -restriction map r_{ij} as follows:

$$(4.1) \quad \begin{aligned} r_{ij} : \mathcal{Riem}_{\text{cap}(\mathbf{p})}^+(S^n) \times \mathcal{Riem}_{\text{cap}(\mathbf{q})}^+(S^n) &\longrightarrow \mathcal{Riem}_0^+(D^n) \times \mathcal{Riem}_0^+(D^n), \\ (g, h) &\longmapsto ((\phi'_{p_i})^*(g|_{D'_{p_i}}), (\phi'_{q_j})^*(h|_{D'_{q_j}})). \end{aligned}$$

We now compose the map r_{ij} with the ordering map ω and the joining map J defined by the formulae 3.6 and 3.5 above to obtain the ij -joining map J_{ij} as:

$$(4.2) \quad \begin{aligned} J_{ij} : \mathcal{Riem}_{\text{cap}(\mathbf{p})}^+(S^n) \times \mathcal{Riem}_{\text{cap}(\mathbf{q})}^+(S^n) &\longrightarrow \mathcal{Riem}_{\text{cap}(\{\mathbf{p} \setminus p_i\} \cup \{\mathbf{q} \setminus q_j\})}^+(S^n), \\ (g, h) &\longmapsto J(\omega(r_{ij}(g, h))). \end{aligned}$$

This construction is illustrated schematically in Figure 5 below where $\mathbf{p} = \{p_0, p_1, p_2, p_3\}$, $\mathbf{q} = \{q_0, q_1, q_2\}$, $i = 1$ and $j = 2$.

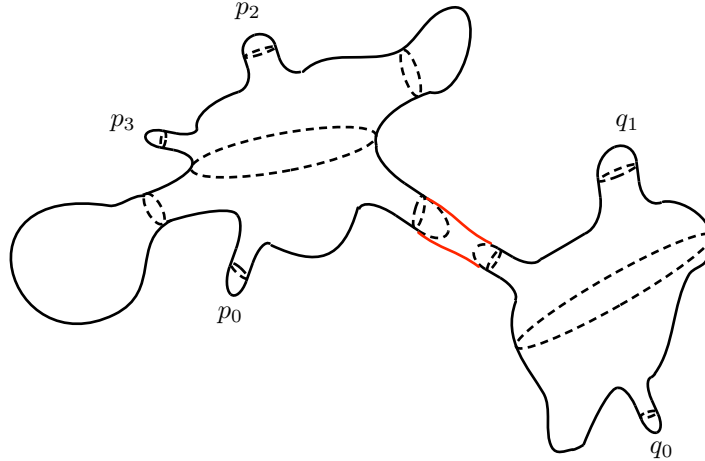


FIGURE 5. The metric $J_{12}(g, h)$ where $\mathbf{p} = \{p_0, p_1, p_2, p_3\}$, $\mathbf{q} = \{q_0, q_1, q_2\}$.

In the case when $\mathbf{p} = \mathbf{q} = \{p_0\}$, this map does not give us the desired homotopy product on $\mathcal{Riem}_{\text{cap}(p_0)}^+(S^n)$ but it gets us most of the way there. We now need to specify an additional object. Henceforth we fix a *base point* $p_0 \in S^n$. Let p_1 and p_2 be two other distinct points on S^n and let $\mathbf{p} = \{p_0, p_1, p_2\}$. Finally let g_3 be any element of $\mathcal{Riem}_{\text{cap}(\mathbf{p})}^+(S^n)$. We now define

a product on $\mathcal{Riem}_{\text{cap}(p_0)}^+(S^n)$ as follows. Consider for each $j = 1, 2$, the map:

$$J_{0j} : \mathcal{Riem}_{\text{cap}(p_0)}^+(S^n) \times \mathcal{Riem}_{\text{cap}(\mathbf{p})}^+(S^n) \longrightarrow \mathcal{Riem}_{\text{cap}(\mathbf{p} \setminus \{p_j\})}^+(S^n),$$

defined as in formula 4.2. If we fix the second input metric as g_3 then for each of $j = 1, 2$ we obtain maps:

$$(4.3) \quad \begin{aligned} J_{0j}^3 : \mathcal{Riem}_{\text{cap}(p_0)}^+(S^n) &\longrightarrow \mathcal{Riem}_{\text{cap}(\mathbf{p} \setminus \{p_j\})}^+(S^n), \\ g &\longmapsto J_{0j}(g, g_3). \end{aligned}$$

Finally, we define a product on $\mathcal{Riem}_{\text{cap}(p_0)}^+(S^n)$ by means of the following continuous map:

$$\begin{aligned} \mu : \mathcal{Riem}_{\text{cap}(p_0)}^+(S^n) \times \mathcal{Riem}_{\text{cap}(p_0)}^+(S^n) &\longrightarrow \mathcal{Riem}_{\text{cap}(p_0)}^+(S^n), \\ (g, h) &\longmapsto J_{02}(h, J_{01}(g, g_3)). \end{aligned}$$

As an illustration, suppose we start with the round metric and assume that p_0, p_1 and p_2 are equidistant along a great circle. Using Lemma 4.3, we push out caps of radius δ_3 (for some sufficiently small $\delta_3 > 0$) at each of the points p_0, p_1 and p_2 to construct the psc-metric g_3 illustrated in Figure 6 below. We call the resulting metric g_3 a *3-cap mine metric* and describe it precisely in the form of a lemma.

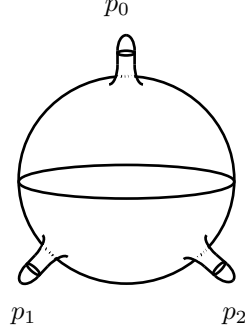


FIGURE 6. The 3-cap mine metric g_3

Lemma 4.5. *Let $n \geq 3$. There is a psc-metric g_3 on S^n which satisfies the following properties.*

- (i) *The metric g_3 agrees with g outside of disjoint neighbourhoods U_0, U_1 and U_2 or the points p_0, p_1 and p_2 respectively.*
- (ii) *For each $i = 1, 2$ or 3 , there are closed disks D_i with $p_i \in D_i \subset U_i$ and diffeomorphisms $\psi_i : D^n \rightarrow D_i$ so that for some $\delta_3 \in (0, 1)$, the metric $\psi_i^*(g_3|_{D_i})$ is an infinitesimal torpedo metric of radius δ_3 .*
- (iii) *The metric g_3 is psc-isotopic to the standard round metric ds_n^2 .*

Proof. This is just an application of Lemma 4.3. □

Figure 7 now depicts the map 4.2 where g_3 is the 3-cap mine metric.

Of course for each triple (p_1, p_2, g_3) where $p_1, p_2 \in S^n \setminus \{p_0\}$ are distinct points and $g_3 \in \mathcal{Riem}_{\text{cap}(\mathbf{p})}^+(S^n)$ (where recall $\mathbf{p} = \{p_0, p_1, p_2\}$), we get a distinct product map $\mu = \mu_{(p_1, p_2, g_3)}$. We will shortly state a lemma which helps clarify the situation up to homotopy. Before this,

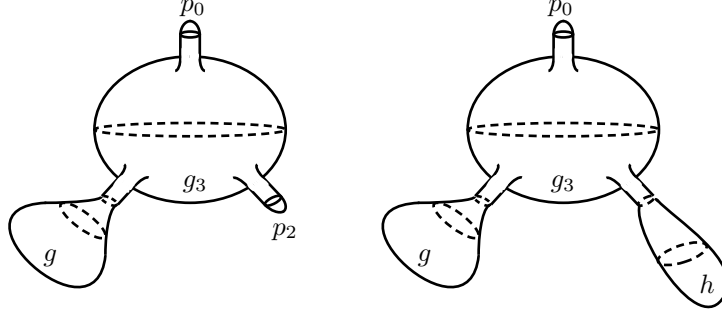


FIGURE 7. The metrics $J_{01}(g, g_3)$ (left) and $\mu(g, h) = J_{02}(h, J_{01}(g, g_3))$ (right)

we recall that two psc-metrics on a smooth manifold M are said to be *psc-isotopic* if there is a path connecting them in the space of psc-metrics $\mathcal{Riem}^+(M)$. We also introduce one other particular metric in $\mathcal{Riem}_{\text{cap}(p_0)}^+(S^n)$. The complement of D_0 in S^n is an open disk whose closure we denote by D' . Thus $D' = \text{closure}(S^n \setminus D_0)$. We now fix a smooth function $\eta_1 \in \mathcal{T}_1^+$ and define $\eta_\delta : [0, \frac{\pi}{2}\delta] \rightarrow [0, \infty)$ by the formula:

$$\eta_\delta(t) = \delta \eta_1\left(\frac{t}{\delta}\right).$$

For each δ , we now have a preferred infinitesimal torpedo metric g^{η_δ} on D^n . Now let $\bar{g}_\delta \in \mathcal{Riem}_{\text{cap}(p_0)}^+(S^n)$ be a metric which satisfies the following properties:

- (i) $\phi_0^* \bar{g}_\delta = g^{\eta_\delta}$
- (ii) The metric \bar{g}_δ restricts on D' to a metric which is isometric to g^{η_δ} .

It will shortly be clear that the metric \bar{g}_1 will play the role of homotopy identity in our multiplication. Before this, we state a lemma concerning the variety of different multiplication maps obtainable by the above process.

Lemma 4.6. *Let $p_1, p_2, p'_1, p'_2 \in S^n \setminus \{p_0\}$. Suppose also that $g_3 \in \mathcal{Riem}_{\text{cap}(\mathbf{p})}^+(S^n)$ and $g'_3 \in \mathcal{Riem}_{\text{cap}(\mathbf{p}')}^+(S^n)$ where $\mathbf{p} = \{p_0, p_1, p_2\}$ and $\mathbf{p}' = \{p_0, p'_1, p'_2\}$. Then the corresponding maps $\mu = \mu_{(p_1, p_2, g_3)}$ and $\mu' = \mu_{(p'_1, p'_2, g'_3)}$ are homotopic if and only if the metrics g_3 and g'_3 are psc-isotopic.*

Proof. It is an immediate consequence of Lemma 4.3 that the process of pushing out a cap while maintaining positive scalar curvature can be done “on the move”, i.e. while the point is moving along a continuous path. Thus the choice of points whether $\{p_1, p_2\}$ or $\{p'_1, p'_2\}$ has no effect up to homotopy.

Now suppose g_3 and g'_3 are psc-isotopic. It is a straightforward consequence of Lemma 4.3 that g_3 and g'_3 can be connected by a continuous path in $\mathcal{Riem}_{\text{cap}(\mathbf{p})}^+(S^n)$. Homotopy equivalence of the maps μ and μ' follows easily. Now suppose μ and μ' are homotopic. Let $\mu_t, t \in I$ be a continuous family of maps

$$\mu_t : \mathcal{Riem}_{\text{cap}(p_0)}^+(S^n) \times \mathcal{Riem}_{\text{cap}(p_0)}^+(S^n) \longrightarrow \mathcal{Riem}_{\text{cap}(p_0)}^+(S^n),$$

where $\mu_0 = \mu$ and $\mu_1 = \mu'$. Consider the path in $\mathcal{Riem}^+(S^n)$ given by $t \mapsto \mu_t(\bar{g}_1, \bar{g}_1)$. This path is now easily deformed, using Lemma 4.3 into a path which connects the metrics g_3 and g'_3 . \square

Given that the space $\mathcal{Riem}^+(S^n)$ is often not path connected, it is evident that there are many non-homotopic possibilities for the map μ . In the next section we will see that in order to make $\mathcal{Riem}_{\text{cap}(p_0)}^+(S^n)$ into a H -space, we will have to restrict our choice of g_3 to metrics which lie in the same path component of the standard round metric.

4.3. The H -Space Theorem. We are now able to state and prove the first of our main results. Let μ be the map defined in 4.2 above with respect to some triple (p_1, p_2, g_3) where as before $p_1, p_2 \in S^n \setminus \{p_0\}$ are distinct points and $g_3 \in \mathcal{Riem}_{\text{cap}(\mathbf{p})}^+(S^n)$.

Theorem 4.7. *Let $n \geq 3$ and let $\mu = \mu_{(p_1, p_1, g_3)}$ be the map given by formula 4.2. In the case when the metric g_3 is psc-isotopic to the round metric ds_n^2 , μ defines a homotopy product on $\mathcal{Riem}_{\text{cap}(p_0)}^+(S^n)$, giving it the structure of a H -space. Furthermore this product is both homotopy commutative and homotopy associative.*

Proof. For convenience take g_3 to be the 3-cap mine metric constructed in the previous section. This is reasonable given Lemma 4.6 and our hypothesis that all choices of g_3 are psc-isotopic to the standard round metric ds_n^2 . We begin by showing that the metric \bar{g}_1 constructed earlier plays the role of homotopy identity. We will show that the map $g \mapsto \mu(g, \bar{g}_1)$ is homotopic to the identity map $g \mapsto g$. The case of $g \mapsto \mu(\bar{g}_1, g)$ is completely analogous. For the most part this will involve a psc-isotopy of the metric g_3 . To help see this we represent, for an arbitrary $g \in \mathcal{Riem}_{\text{cap}(p_0)}^+(S^n)$, the element $\mu(g, \bar{g}_1)$ in Figure 8 below. We will now construct a deformation of the map $g \mapsto \mu(g, \bar{g}_1)$ to the identity map. This

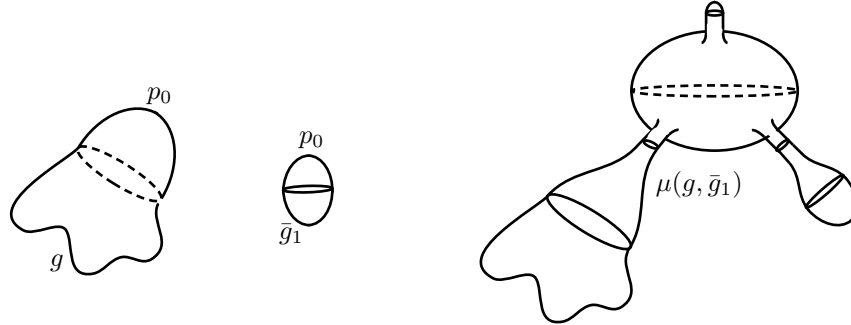


FIGURE 8. The metrics g, \bar{g}_1 and $\mu(g, \bar{g}_1)$

is precisely the map $g \mapsto J_{01}(g, J_{02}(\bar{g}_1, g_3))$. We begin by constructing a psc-isotopy of the metric $J_{02}(\bar{g}_1, g_3)$ which we depict in Figure 9. Firstly, the part of this metric consisting of $\bar{g}_1|_{D'}$ and its connecting metric can easily be adjusted by an isotopy to a standard and then infinitesimal torpedo by means of the results of section 1 of [15]. The resulting metric is the first image in Figure 9. This psc-isotopy induces a homotopy to the map $g \mapsto J_{01}(g, g_3)$. Then, by lemma 4.5 we contract the cap at p_2 on the mine metric and isotopy (by a simple rotation) the remaining metric to one where the cap at p_0 is antipodal to the connecting metric with g . This metric is depicted in the second image in Figure 9. This metric then easily contracts, via the results of chapter 1 of [15], to the one shown in the third image in Figure 9 which is a standard δ_3 -torpedo metric on $S^n \setminus D_1$ and is connected to $g|_{D'}$ via a

connecting metric. This induces a homotopy to the map $g \mapsto J_{01}(g, g_{Dtor}^n(\delta_3))$ where g_{Dtor}^n is the metric obtained on S^n by gluing a pair of torpedo disk metrics along their boundary in the standard way (the “double” torpedo metric). The neck of this torpedo can then be contracted down inducing a homotopy of the map $g \mapsto J_{01}(g, \bar{g}_{\delta_3})$. We finally complete the homotopy by replacing δ_3 with $(1-t)\rho(g) + t\delta_3$ in the formula for $J_{01}(g, \bar{g}_{\delta_3})$. This isotopes the metric $J_{01}(g, \bar{g}_{\delta_3})$ back to the metric g depicted again in the last image of Figure 9.

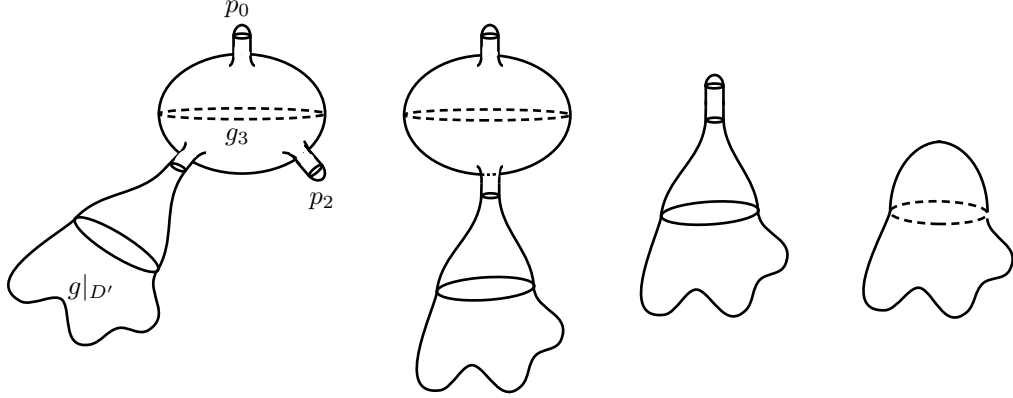


FIGURE 9. An isotopy of the metric $J_{02}(\bar{g}_1, g_3)$ back to g

Homotopy commutativity follows immediately from Lemma 4.6. In general, just choose $p'_1 = p_2$ and $p'_2 = p_1$ in the statement of that lemma. In the case when g_3 is the 3-cap mine metric, this is most easily induced by continuous rotation of the sphere which swaps p_1 with p_2 as shown in Figure 10 below.

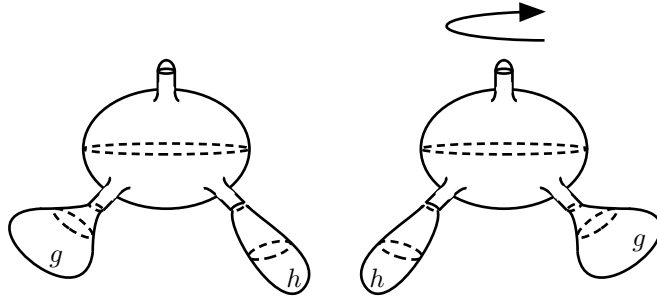


FIGURE 10. The metrics $\mu(g, h)$ and $\mu(h, g)$.

It remains to show homotopy associativity. This is similar to the proof of homotopy commutativity in that it involves moving the metrics of g_3 along a closed bounded arc on a psc-manifold, adjusting the radius if necessary. Again we utilise Lemma 4.3 to move the metrics continuously and maintain positivity of the scalar curvature. In particular, this means that for any $g, h, h' \in \text{Riem}_{\text{cap}(p_0)}^+(S^n)$ the metric $\mu(\mu(g, h), h')$ is psc-isotopic to $\mu(g, \mu(h, h'))$ as shown in Figure 11 below.

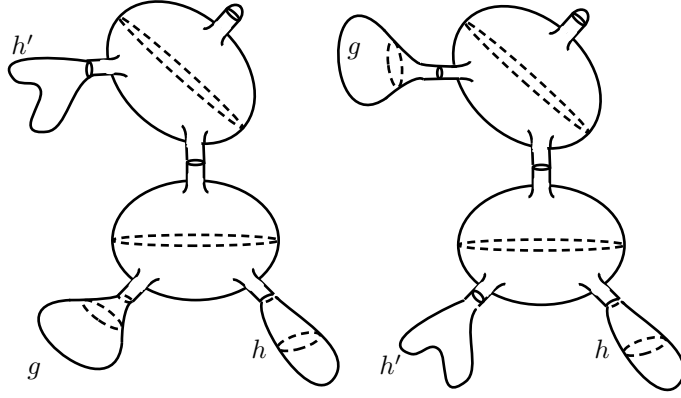


FIGURE 11. The metrics $\mu(\mu(g, h), h')$ and $\mu(g, \mu(h, h'))$.

With this in mind, let g_3 and g'_3 be two copies of the same intermediary metric with cap points $\{p_0, p_1, p_2\}$ and $\{p'_0, p'_1, p'_2\}$ respectively. Obviously p'_0 is identified with the base point $p_0 \in S^n$. We now fix a psc-isotopy $g(t)$ which moves $g(0) = J_{01}(g_3, g'_3)$ to the metric $g(1)$ obtained by swapping the p_1 cap of g_3 with the p'_2 cap of g'_3 . Now consider the map $(g, h, h') \mapsto \mu_0(g, h, h') = \mu(\mu(g, h), h')$. Viewing the metric $J_{01}(g_3, g_3)$ as the metric $J_{01}(g_3, g'_3)$ and then replacing it by $g(t)$ induces a homotopy of maps μ_t between μ_0 which is defined above and μ_1 defined by $\mu_1(g, h, h') = \mu(g, \mu(h, h'))$. \square

Corollary 4.8. *For $n \geq 3$, the group $\pi_1(\text{Riem}^+(S^n), ds_n^2)$ is Abelian.*

Proof. This is a well-known fact about H -spaces and so we will be terse. Suppose Z is a H -space with multiplication μ and homotopy identity $e \in Z$. Let $\alpha, \beta : [0, 1] \rightarrow Z$ be loops based at e representing classes of $\pi_1(Z, e)$. Now consider the map $F : [0, 1] \times [0, 1] \rightarrow Z$ which is defined $F(s, t) = \mu(\alpha(s), \beta(t))$. Depending on one's choice of parameterisation, the restriction of F firstly to the left and bottom sides of the square and secondly to the top and right sides of the square gives rise to maps which are respectively homotopy equivalent to the loop concatenations $\alpha \circ \beta$ and $\beta \circ \alpha$. Constructing the appropriate homotopy is then a straightforward exercise. \square

5. THE ITERATED LOOP SPACE THEOREM.

We now turn our attention to the second of our main results. This involves showing that $\text{Riem}^+(S^n)$ is weakly homotopy equivalent to an n -fold loop space when $n \geq 5$. The problem of recognising when a H -space is an iterated loop space is an old problem in Algebraic Topology; see [9] for a thorough account of this story. A key step in tackling this problem was the discovery by Boardman and Vogt that an n -fold loop space admits an action of the *operad of little n -dimensional disks*. Before explaining what this means, we should point out that we are interested in a converse to this proposition. That is, given a space which admits such an action, is it an n -fold loop space? Under reasonable conditions such a converse holds. This is the subject of a theorem of Boardman, Vogt and May, which we will state shortly and which allows us to prove our main result: Theorem 5.5. Before that however, we need to discuss the aforementioned operad of little disks.

5.1. The operad of little n -dimensional disks. We will not enter into a general discussion of *operads* as this is unnecessary here. For more on this subject, see [9]. We will simply provide a brief summary of a discussion from chapter 4.3 of [9]. For $n \geq 1$ (although in our case we will only be interested in $n \geq 3$) we recall that D^n denotes the standard closed unit disk in \mathbb{R}^n . Let $j \geq 0$ be an integer. We denote by D_j^n the ordered j -tuple of closed round disks which are contained in the interior of D^n . To ease notation we will fix an n and simply write D_j instead of D_j^n . Each element of D_j is therefore an ordered j -tuple of little disks. There is of course an obvious action on D_j of the permutation group Σ_j .

Notice that for each little disk, there is a canonical homeomorphism which identifies it with the larger unit disk D^n i.e. shrink D^n and translate. This allows us to construct the following “fitting” map. Consider the product space $D_k \times D_{j_1} \times D_{j_2} \times \cdots \times D_{j_k}$. Suppose $\{d_k, d_{j_1}, d_{j_2}, \dots, d_{j_k}\}$ is an element of this space. The first component d_k consists of k ordered little disks in D^n . By appropriately shrinking and translating the standard unit disk we may “fit” each of the elements d_{j_r} into the corresponding r -th little disk of d_k . This is shown for a particular example when $k = 2, j_1 = 3$ and $j_2 = 2$ in Figure. 12 below. The result is an element of $D_{j_1+j_2+\dots+j_k}$ which we denote $d_k(d_{j_1}, \dots, d_{j_k})$. We summarise the *fitting map*, which we denote γ , as follows:

$$\begin{aligned} \gamma : D_k \times D_{j_1} \times D_{j_2} \times \cdots \times D_{j_k} &\longrightarrow D_{j_1+j_2+\dots+j_k} \\ (d_k; (d_{j_1}, d_{j_2}, \dots, d_{j_k})) &\longmapsto d_k(d_{j_1}, d_{j_2}, \dots, d_{j_k}). \end{aligned}$$

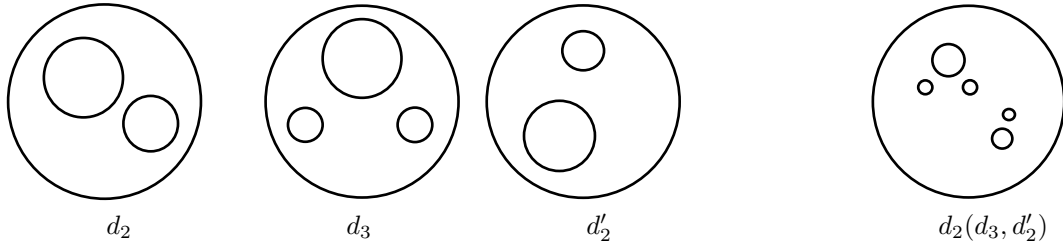


FIGURE 12. The fitting map γ in action

The fitting maps satisfy *strict associativity* and *permutation equivariance* conditions. More precisely,

$$d_k[d_{j_1}(d_{11}, \dots, d_{1j_1}), \dots, d_{j_k}(d_{k1}, \dots, d_{kj_k})] = [d_k(d_{j_1}, \dots, d_{j_k})](d_{11}, d_{12}, \dots, d_{2j_1}, \dots, d_{kj_k}),$$

and

$$d_k^\sigma(d_{j_1}^{\sigma_1}, \dots, d_{j_k}^{\sigma_k}) = d_k(d_{j_1}, \dots, d_{j_k})^{\tilde{\sigma} \circ (\sigma_1 + \sigma_2 + \dots + \sigma_k)},$$

where $d_k^{\sigma_k}$ is the effect of permutation on d_k by $\sigma \in \Sigma_k$ and $\tilde{\sigma}$ is a σ -block permutation. Finally we define the space \mathbb{D}_n as:

$$\mathbb{D}_n = \bigcup_{j \geq 0} D_j^n.$$

This, along with the fitting maps γ which satisfy the above properties, is known as the *operad of little n -disks*. Now suppose W is a topological space. We describe W as a \mathbb{D}_n -space if for

each integer $k \geq 0$ there are actions

$$\theta : D_k \times W^k \longrightarrow W,$$

so that the following conditions hold.

- (1) $\theta(d_k^\sigma, (w_1, w_2, \dots, w_k)) = \theta(d_k, (w_{\sigma^{-1}(1)}, \dots, w_{\sigma^{-1}(k)}))$ for all $\sigma \in \Sigma_k$, $d_k \in D_k$ and $(w_1, \dots, w_k) \in W^k$.
- (2) The following diagram commutes.

$$\begin{array}{ccccc} D_k \times D_{j_1} \times D_{j_k} \times W^j & \xrightarrow{\gamma \times Id} & D_j \times W^j & \xrightarrow{\theta} & W \\ \downarrow \text{shuffle} & & & & \downarrow Id \\ D_k \times D_{j_1} \times W^{j_1} \times \dots \times D_{j_k} \times W^{j_k} & \xrightarrow{Id \times \theta^k} & D_k \times W^k & \xrightarrow{\theta} & W \end{array}$$

We close this section by stating a theorem due to Boardman, Vogt and May which will be crucial in proving Theorem 5.5. This is Theorem 4.25 from [9].

Theorem. (Boardman, Vogt and May) [9] *If a \mathbb{D}_n -space is group-like (i.e. π_0 is a group), then it is weakly homotopy equivalent to an n -fold loop space.*

The remainder of the paper will involve showing that $\mathcal{Riem}_{\text{cap}(p_0)}^+(S^n)$ satisfies the hypotheses of this theorem.

5.2. Showing that $\mathcal{Riem}_{\text{cap}(p_0)}^+(S^n)$ is a \mathbb{D}_n -space. In order to show that $\mathcal{Riem}_{\text{cap}(p_0)}^+(S^n)$ is a \mathbb{D}_n -space we must first construct a sort of an embedding of the operad of little n -disks into the space $\mathcal{Riem}_{\text{cap}(p_0)}^+(S^n)$. The elements of \mathbb{D}_n will be realised as psc-metrics which in a strong sense generalise the 3-cap mine metric constructed earlier. We begin with a technical observation. Consider the round hemispherical metric of radius 1 on the disk D^n . In spherical coordinates this takes the form:

$$dt^2 + \sin^2 t ds_{n-1}^2,$$

where the radial distance coordinate t satisfies $0 < t \leq \frac{\pi}{2}$. Thus, each $t_0 \in (0, \frac{\pi}{2}]$ corresponds to a disk centered at the origin with a piece of a round metric specified by the formula above with $t \in (0, t_0]$ as shown below in Figure. 13.

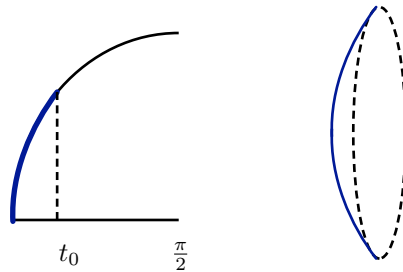


FIGURE 13. Each t_0 (left) specifies a piece of the round metric (right).

In [15] we show in great detail that, using the work of Gromov and Lawson [5], for each $t_0 \in (0, \frac{\pi}{2}]$ there is a real number $L \in [0, \infty)$ and a function $\gamma_{t_0} : (-L, t_0] \rightarrow (0, \infty)$ which satisfies the following properties:

- (1) Near t_0 , γ_{t_0} agrees with \sin .
- (2) For some $\delta > 0$, the restriction of γ_{t_0} to the interval $(-L, -L + \frac{\pi}{2}\delta]$ is an element of \mathcal{T}_δ^+ , i.e. is an infinitesimal torpedo curve of radius δ .
- (3) The metric $dt^2 + \gamma_{t_0}^2(t)ds_{n-1}^2$, where $t \in (-L, t_0]$ has positive scalar curvature.

The graph of such a function is described in Figure. 14 below. From work in [15], we now

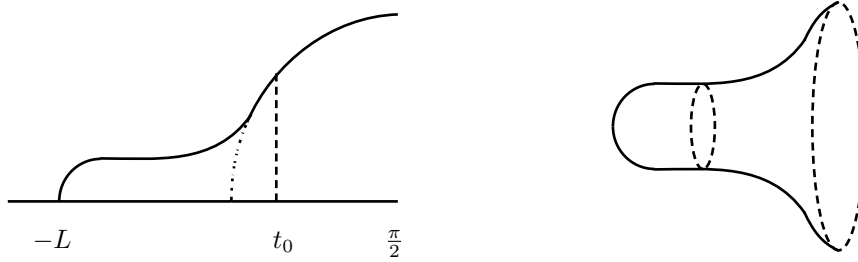


FIGURE 14. The curve γ_{t_0} (left) and the resulting psc-metric (right).

have the following lemma.

Lemma 5.1. *There is a continuous family of functions $\gamma_{t_0} : (-L, t_0] \rightarrow (0, \infty)$ satisfying the above conditions and parameterised by $t_0 \in (0, \frac{\pi}{2}]$.*

Proof. This follows easily from Theorem of [15]. □

Recall that S^n represents the standard unit round sphere centred at the origin in \mathbb{R}^{n+1} . We denote by D_+ and D_- the closed disks $\{x \in \mathbb{R}^{n+1} : x \cdot x = 1 \text{ and } x_{n+1} \geq 0\}$ and $\{x \in \mathbb{R}^{n+1} : x \cdot x = 1 \text{ and } x_{n+1} \leq 0\}$ respectively, i.e. the closed northern and southern hemispheres of S^n . For convenience we will assume that p_0 is the point $(0, 0, \dots, 1) \in \mathbb{R}^{n+1}$, i.e. the north pole of the sphere S^n . Starting with the standard round metric we make an adjustment in a neighbourhood of p_0 , exactly as in the construction of the 3-cap mine metric in section 4.2. From Lemma 4.5, the resulting psc-metric denoted g_{op} and depicted in Figure. 15 below, satisfies the following properties:

- (i) There is a neighbourhood U_0 contained entirely inside the interior of the northern hemisphere D_+ of S^n so that outside of U_0 , the metric g_{op} agrees with ds_n^2 .
- (ii) There is a closed disk neighbourhood $D_0 \subset U_0$ of p_0 and a diffeomorphism $\phi_0 : D^n \rightarrow D_0 \subset U_0$ so that $\phi_0^* g_{op} \in \mathcal{T}_{\text{Riem}}^+$.
- (iii) The metric g_{op} is psc-isotopic to the standard round metric ds_n^2 .

We will now prove the following lemma.

Lemma 5.2. *For $n \geq 3$, the space $\text{Riem}_{\text{cap}(p_0)}^+(S^n)$ is a \mathbb{D}_n -space.*

Proof. We begin by identifying the standard disk D^n with the southern hemisphere D_- of the standard sphere S^n in the obvious way by the map $x \mapsto (x, \sqrt{1 - x \cdot x})$. This southern hemisphere should now be thought as the southern hemisphere of the metric g_{op} described

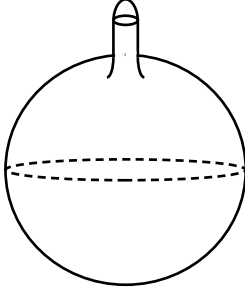


FIGURE 15. The metric g_{op} .

above. In particular, this means that the element of \mathbb{D}_n containing no little disks is identified with g_{op} . Non-trivial elements of \mathbb{D}_n can now be identified with elements of $\mathcal{Riem}_{cap(p_0)}^+(S^n)$ via the following steps (and described in Figure. 16 below):

- (1) Suppose $d(p, \epsilon)$ is a little disk on D^n with centre p and radius ϵ . We identify such a disk with the closed ball on the southern hemisphere of S^n which is centred at the point $(p, \sqrt{1 - p \cdot p})$ and of radius $\arcsin(\epsilon)$, this time with respect to the round metric on the hemisphere. Again this hemisphere should be thought of as part of g_{op} . It is a trivial exercise to see that k disjoint closed balls on the flat disk will be sent to k disjoint closed balls on the hemisphere by this identification.
- (2) Now, for each such closed ball of radius t_0 on the round hemisphere, Lemma 5.1 specifies a unique adjustment of the metric g_{op} inside this ball, determined by the curve γ_{t_0} . The adjustment involves the pushing out of a cap around the point $(p, \sqrt{1 - p \cdot p})$. Importantly, once all the adjustments are made, the resulting metric still has positive scalar curvature.

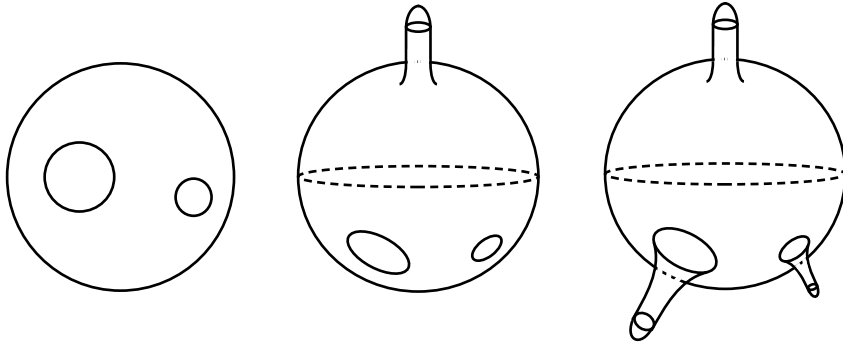


FIGURE 16. Identifying an element of \mathbb{D}_n (left) with an element of $\mathcal{Riem}_{cap(p_0)}^+(S^n)$ (right).

This identification describes an injection of the elements of the little n -disks operad \mathbb{D}_n into $\mathcal{Riem}_{cap(p_0)}^+(S^n)$. It is easy to check that the fitting maps and their various properties are preserved by this injection. We are now in a position to describe the action θ of \mathbb{D}_n on $\mathcal{Riem}_{cap(p_0)}^+(S^n)$. Given an element $d_k \in \mathbb{D}_n$, we will denote the corresponding psc-metric in $\mathcal{Riem}_{cap(p_0)}^+(S^n)$, given by the association above, by $g(d_k)$ also. Recall that d_k consists of k

ordered little disks and thus the psc-metric $g(d_k)$ has k ordered caps. For simplicity we will say these caps are at points $p_1, p_2 \cdots p_k \in D_- \subset S^n$. Using the ij -joining map described in 4.2, we define θ as follows:

$$\begin{aligned} \theta : D_k \times \mathcal{Riem}_{\text{cap}(p_0)}^+(S^n)^k &\longrightarrow \mathcal{Riem}_{\text{cap}(p_0)}^+(S^n) \\ (d_k; (g_1, g_2, \cdots g_k)) &\longmapsto J_{kk}(J_{(k-1)(k-1)} \cdots J_{22}(J_{11}(g(d_k), g_1), g_2) \cdots g_{k-1}), g_k). \end{aligned}$$

Effectively this simply means attaching the i -th metric to the i -th cap using the joining map described earlier. We schematically represent this in the case when $k = 3$ in Figure 17 below. Checking that θ satisfies the appropriate equivariance properties to make $\mathcal{Riem}_{\text{cap}(p_0)}^+(S^n)$ a \mathbb{D}_n -space is now an easy combinatorial exercise. \square

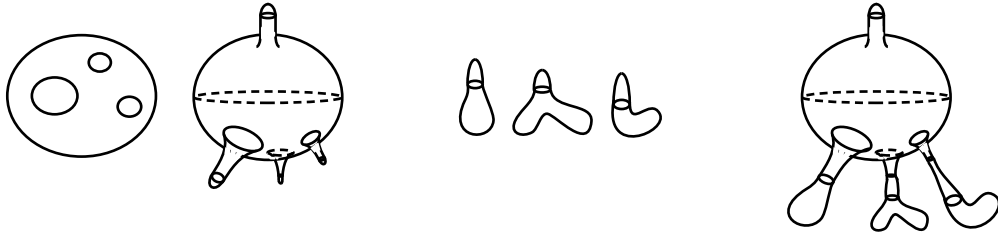


FIGURE 17. An example of the operad action when $k = 3$

5.3. Showing that π_0 is a group. In this section we wish to show that $\pi_0(\mathcal{Riem}_{\text{cap}(p_0)}^+(S^n))$ is a group under the operation induced by the H -space connected sum operation defined in section 4. Initially, we will show that $\pi_0(\mathcal{Riem}^+(S^n))$ is a group under the operation induced by regular connected sum. As $\mathcal{Riem}_{\text{cap}(p_0)}^+(S^n)$ and $\mathcal{Riem}^+(S^n)$ are homotopy equivalent, this gets us most of the way there.

Recall that $\pi_0(\mathcal{Riem}^+(S^n))$ is the set of path components of $\mathcal{Riem}^+(S^n)$. The notion of psc-metrics lying in the same path component is of course an equivalence relation on the set of psc-metrics and, in the case of the space $\mathcal{Riem}^+(S^n)$, this relation is called *psc-isotopy*. In other words, two metrics in $\mathcal{Riem}^+(S^n)$ are *psc-isotopic* if they lie in the same path component. A related notion, which we will make use of shortly is the notion of psc-concordance. In the case of metrics $g_0, g_1 \in \mathcal{Riem}^+(S^n)$, we say that g_0 and g_1 are *psc-concordant* if there is a psc-metric \bar{g} on $S^n \times I$ which near $S^n \times \{0\}$ takes the form of a product $g_0 + dt^2$ and near $S^n \times \{1\}$ takes the form $g_1 + dt^2$. Again, psc-concordance is an equivalence relation on the set of psc-metrics $\mathcal{Riem}^+(S^n)$.

It is a well known fact that metrics which are psc-isotopic are psc-concordant; see Lemma 1.3 from [15] for example. Recent work by Botvinnik in [1] has shown that, under reasonable hypotheses, the converse is true. In particular, the following theorem is a case of Theorem B. from [1].

Theorem. (Botvinnik) [1] *Let $g_0, g_1 \in \mathcal{Riem}^+(S^n)$. When $n \geq 5$, g_0 is psc-isotopic to g_1 if and only if g_1 is psc-concordant to g_0 .*

The above theorem will play an important role in the proof of our main result. It is worth noting that the hypothesis that n be at least five cannot be removed as the above result fails

to be true when $n = 4$. In the case when $n = 3$, it is demonstrated by Marques in [10] that the space $\mathcal{Riem}^+(S^n)$ is path connected, and so the theorem holds for trivial reasons. We now return to the problem of equipping $\pi_0(\mathcal{Riem}^+(S^n))$ with a group structure. The case of $\mathcal{Riem}_{\text{cap}(p_0)}^+(S^n)$ is then taken care of as a corollary.

Lemma 5.3. *For $n = 3$ or $n \geq 5$, the set $\pi_0(\mathcal{Riem}^+(S^n))$ is a group under the operation induced by connected sum of metrics.*

Proof. Corollary 1.1 of [10] states that the space $\mathcal{Riem}^+(S^n)$ is path-connected when $n = 3$. We therefore concentrate on the case when $n \geq 5$. The group operation is of course induced by the Gromov-Lawson connected sum construction on psc-metrics. In the case of two psc-metrics g_0 and g_1 on S^n , we may form a new psc-metric $g_0 \# g_1$ by removing disks from (S^n, g_0) and (S^n, g_1) and connecting the resulting disks with a metric $S^{n-1} \times I$ equipped with an appropriate connecting psc-metric a la Gromov and Lawson. At the level of psc-metrics this is not a well-defined operation, however once we descend to the level of isotopy classes these problems disappear. This is an easy corollary of Lemma 4.3. Hence we have a well defined binary operation on $\pi_0(\mathcal{Riem}^+(S^n))$.

Verifying that the various group axioms hold is mostly straightforward. In particular, it is clear that the class containing the standard round metric $[ds_n^2]$ is the identity. The only difficulty lies in verifying that each element has an inverse. To see this we briefly return to the notion of psc-concordance. Gajer in [4] shows that the set of concordance classes of $\mathcal{Riem}^+(S^n)$, which we denoted $\pi_0^c(\mathcal{Riem}^+(S^n))$, forms a group also under the operation induced by connected sum. We won't reprove it here but it is worth briefly recounting Gajer's method for showing that each element has an inverse. Given a psc-metric g on S^n which represents a particular concordance class, equip $S^n \times I$ with the standard product $g + dt^2$. Let $p \in S^n$ be any point. Consider the this arc $\{p\} \times I$ in $S^n \times I$. Using Lemma 4.3 slicewise, one can easily adjust the metric $g + dt^2$ in a neighbourhood of this arc to obtain a psc-metric $g' + dt^2$ so that near $\{p\} \times I$, $g' + dt^2 = g_{\text{tor}}^n(\delta) + dt^2$ for some $\delta > 0$. Recall that $g_{\text{tor}}^n(\delta)$ is the standard torpedo metric of radius δ on the disk. Next we use the Gromov-Lawson method to push out a torpedo cap away from this neighbourhood and preserve positive scalar curvature. By removing the cap part, also removing the previously constructed "cylinder of caps" and finally smoothing out the inevitable corners, we are left with a manifold which is topologically a cylinder $S^n \times I$ but with a very different metric; see Figure 18. At one end we have a standard round metric of radius δ . At the other end we have the psc-metric $g \# g^{-1}$ obtained by taking a connected sum of g and g^{-1} . Here g^{-1} is isometric to g but via an orientation reversing isometry. In Theorem 2.2 of [15], we show in great detail how to adjust a psc-metric on a manifold with boundary in precisely this situation in order to obtain a psc-metric which has a product structure near the boundary. On performing such an adjustment we obtain a psc-concordance between $\delta^2 ds_n^2$ and $g \# g^{-1}$ and thus between ds_n^2 and $g \# g^{-1}$. Thus the classes containing g and g^{-1} are inverses in the group $\pi_0^c(\mathcal{Riem}^+(S^n))$.

To show that g and g^{-1} represent inverse elements in $\pi_0(\mathcal{Riem}^+(S^n))$, we need only show that as well as being psc-concordant, the round metric and the connected sum of g and g^{-1} are also psc-isotopic. That these metrics are indeed psc-isotopic when $n \geq 5$ follows of course from the aforementioned theorem of Botvinnik, Theorem B. of [1].

□

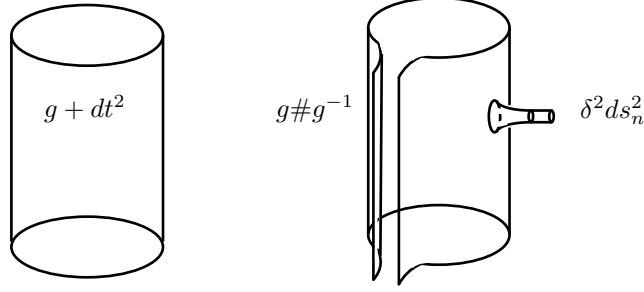


FIGURE 18. The cylinder $g + dt^2$ (left) and the metric which gives rise, after adjustment, to the concordance between $g \# g^{-1}$ and $\delta^2 ds_n^2$ (right)

Corollary 5.4. *For $n = 3$ or $n \geq 5$, the set $\pi_0(\mathcal{Riem}_{\text{cap}(p_0)}^+(S^n))$ is a group under the operation induced by the H -space connected sum operation.*

Proof. The inclusion $\mathcal{Riem}_{\text{cap}(p_0)}^+(S^n) \subset \mathcal{Riem}^+(S^n)$ induces a bijection between the sets $\pi_0(\mathcal{Riem}_{\text{cap}(p_0)}^+(S^n))$ and $\pi_0(\mathcal{Riem}^+(S^n))$. The corollary then follows from the fact that the intermediary metric g_3 from the H -space connected sum is isotopic to the standard round metric. This means that any H -space connected sum is easily deformed by isotopy to a regular connected sum. Hence the two operations behave in the same way with regard to path components. \square

5.4. The Loop Space Theorem. By combining the results of the previous sections we can now prove the following theorem, our second main result.

Theorem 5.5. *For $n = 3$ or $n \geq 5$, the space of positive scalar curvature metrics on the n -dimensional sphere, $\mathcal{Riem}^+(S^n)$, is weakly homotopy equivalent to an n -fold loop space.*

Proof. We know from Lemma 4.4 that when $n \geq 3$ and for any $p_0 \in S^n$, the spaces $\mathcal{Riem}^+(S^n)$ and $\mathcal{Riem}_{\text{cap}(p_0)}^+(S^n)$ are homotopy equivalent. It is therefore enough to prove the theorem for $\mathcal{Riem}_{\text{cap}(p_0)}^+(S^n)$. Following the theorem of Boardman, Vogt and May (Theorem 4.25 in [9]) stated at the end of section 5.1, we need only show that $\mathcal{Riem}_{\text{cap}(p_0)}^+(S^n)$ is a \mathbb{D}^n -space and that $\pi_0(\mathcal{Riem}_{\text{cap}(p_0)}^+(S^n))$ is a group under the operation induced by the H -space connected sum. The first of these is done when $n \geq 3$ in Lemma 5.2 while the second is done when $n = 3$ or when $n \geq 5$ in Corollary 5.4. \square

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